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Asymptotic analysis of time singularities for a class of time-dependent Hamiltonians

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Abstract. We analyse the local and global structure of time singularities for a class of quasi-integrable Hamiltonian systems in the Arnold–Liouville sense. We show that there is good agreement between the numerically observed local behaviour of the solutions and the perturbative scheme we produce using asymptotic approximations of the solution around the singularities. We also prove the convergence of the Psi-series associated to the movable singularities of the systems considered. We also propose a simple model in order to analyse the global structure of the singularities in the directions of exponential growth of the potential in time.

1. Introduction

In this paper we consider quasi-integrable Hamiltonian systems with Hamiltonians of the following form:

$$\mathcal{H}(q, p, t) = \frac{p^2}{2} + \mathcal{V}(q) + \epsilon\mathcal{R}(q, t) \tag{1.1}$$

where $\mathcal{V}$ and $\mathcal{R}$ are algebraic in q and $\mathcal{R}$ is entire in t. We also suppose that $\deg \mathcal{V} \geqslant \deg \mathcal{R}$. q and p are chosen to be complex.

The general solution of the nonlinear time-dependent equations associated to (1.1) may be locally represented as psi series [11], possessing movable singularities. In the case of the Duffing equation (see [3, 4, 8, 9, 18]) the appearance of clusters of movable singularities has been evidenced numerically near a movable critical point. In particular, these authors have considered the psi series associated to the general integrals of the Duffing system in order to give a justification of the numerical behaviour. Psi series have also been considered in the case of other systems (see [6, 13, 16, 17]).

We are interested here in the asymptotic behaviour in time near a movable singularity with arbitrary degree of $\mathcal{V}$ in (1.1).

When $\epsilon = 0$ the global structure of the movable singularities of $q(t)$ is, in general, non-trivial, provided the $\deg \mathcal{V}$ is sufficiently high. In fact, when $\deg \mathcal{V} \leqslant 4$ the movable time singularities are non-critical poles (the system is actually also Painlevé-integrable [7, 12, 15] in this case) and the global structure of the Riemann sheets may be described quite easily. When $\deg \mathcal{V} > 4$, the movable singularities are algebraic in nature and, in the generic case critical; that is, they produce an infinite sheeted Riemann foliation in the large which cannot actually be uniformly described. In both cases, the singularities, whether critical or not, are isolated.

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When we add a time-analytic perturbation ($\epsilon \neq 0$), the movable time singularities of $q(t)$ still have locally the same polar or algebraic dominant part as in the corresponding unperturbed case. This follows from the assumption $\deg \mathcal{V} \geqslant \deg \mathcal{R}$. Their nature, instead, is logarithmic for a generic time-analytic perturbation, as can be shown by expressing locally $q(t)$ via a psi series.

Numerically, if we start to integrate the differential equations associated with (1.1), along paths which turn around one of such singularities, we continuously change the Riemann sheet and observe new singularities arising and disappearing at each turn (see figures 3(*b*), 4(*b*), 5(*b*), 6(*a*) and 7(*a*)).

In order to explain such a behaviour we make a certain ansatz about the dominant terms in the psi series associated to $q(t)$ in the neighbourhood of a movable critical singularity. We then use the Painlevé α-technique where, in our approximation, α is time-dependent, and we give a constructive asymptotic approximation of the solution.

Such asymptotic approximation can be interpreted as the summation of the psi series taking its terms in an appropriate order. So the main problem is whether such a procedure is justified, that is whether the psi series converges absolutely. Indeed, for simplicity of notation, we consider a subclass of systems of the form (1.1) and prove that such series converge absolutely in convenient sectors of the complex time plane, so that the method proposed here is self-consistent. We do not consider here the optimal convergence ratio, but we are convinced that the maximal radius of convergence is connected with the appearance of the first singularity of the 'cluster'.

Concerning the convergence of psi series solution [10, 11, 19], recently, Melkonian and Zypchen [14] have shown the convergence of the psi series in the case of the Duffing equation and the Lorenz system.

In particular, the asymptotic analysis proposed here shows that the Hamiltonian system (1.1) may be conjugated with a time-independent one via a transformation which conjugates the local singularity structure of (1.1) to the global singularity structure of the corresponding time-independent system. Of course, this allows one to predict the local singularity structure quite accurately only when the movable singularities of the underlying time-independent system are non-critical ($\deg \mathcal{V} \leqslant 4$) or produce only finite branching (like in the second example of section 4).

We also consider numerically the time singularity structure when the time-dependent perturbation in (1.1) grows exponentially in a certain complex time direction. In these cases, if we numerically integrate the equations of motion along such particular time direction, we see the appearance of singularities which look as if they accumulate on the integration path, forming a sort of 'barrier' for the singularities (see figures 1, 2, 3(*a*), 4(*a*), 5(*a*), 8 and 9). Such 'peaks' have already been observed in the case of the Duffing system with a time-dependent term of the form $\sin(\omega t)$—which is a particular example in the class we consider here—in [4, 8, 18].

We are convinced that such peaks are due to the exponential growth of the potential along the integration path. In order to support this conjecture, we consider here the simplest model with $\deg \mathcal{V} = 3$, which is integrable in suitable coordinates. In particular, using this model, we can prove that the singularities approach the integration path at a sufficiently low speed. This allows for analytic continuation of the solution in the direction in which the potential explodes.

Finally, we have also started a numerical investigation of the structure of such chimneys for a different example with $\deg \mathcal{V} = 3$, in order to give lower bounds to the rate of approximation of the singularities to the integration path.

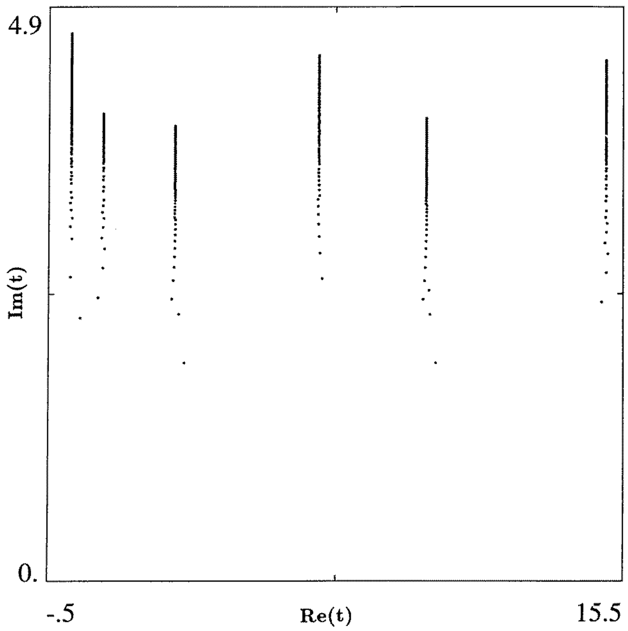


Figure 1. We plot the global singularity structure in time for the differential equation $\ddot{q} = q - (1 + \epsilon_1 \exp(bt)/(1 + \exp(bt)))(1 + \epsilon_2 \sin(\omega t))q^2$, where $\epsilon_1 = 0.01$, $\epsilon_2 = 0.001$, $b = 0.3$, $\omega = 2\pi$, $q(0) = 0.5$, $\dot{q}(0) = 0.0$, along path starting from $t = 0$ and moving parallel to real and imaginary axes. The singularities are identified as double poles by ATOMFT; notice that the perturbation is asymptotically periodic in t.

Numerical and asymptotic analyses for large times give compatible bounds on the rate of approximation of the singularities to their asymptotes. In particular, in the examples considered here, the singularities approach the integration path roughly at n^{-1} speed, where n is the index position of the singularity. So the singularities approach the path of integration accumulating on it in a way which could allow for infinite time-analytical continuation of the solutions in the directions of exponential growth of the potential. We think that the rate of accumulation of singularities depends on $\deg \mathcal{V} \neq 3$.

In section 2 and in appendix A, we present our asymptotic model for the local singularity clusters around a movable critical point. In section 3 and in appendix B, we prove the convergence of the psi series. In section 4 and in appendix C, we present an example and apply the perturbative asymptotic techniques to it.

In section 5 we consider the time singularity structure for large times in the directions of exponential growth of the potential; we present an asymptotic model for a class of examples and compare it with numerical simulations.

In section 6 we present our conclusions from the analysis in the previous sections.

2. The Painlevé α-method and an asymptotic approximation near the movable singularities

In this section we consider local approximations of the perturbed differential equation associated to the Hamiltonian (1.1) defined in the introduction

$$\ddot{q} = -\mathcal{V}'(q) + \epsilon \mathcal{R}'(q, t)$$

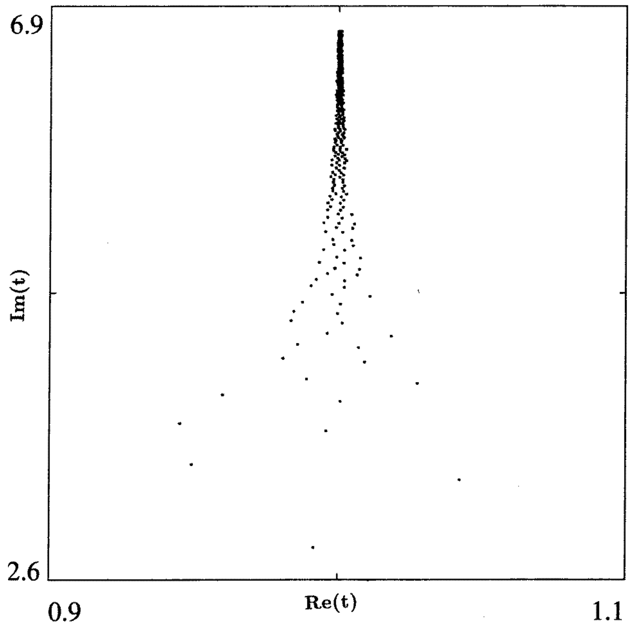


Figure 2. Same as in figure 1, for the differential equation $\ddot{x} = x(\epsilon \sin(\pi t) - 1)x + x^4$ where $\epsilon = 0.01$, $x(0) = 0.5$, $\dot{x}(0) = 0$. and the integration path is along the real time axis up to $t = 1.0$ and then vertical. The singularities are recognized as algebraic of order $-2/3$ by the program and lie on the same Riemann sheet.

where a prime denotes differentiation with respect to the q variable. Since our analysis is local, we may rescale the time variable and consider the case where

$$\ddot{q} = q^l + \mathcal{W}(q, t) = q^l + \sum_{j=0}^{l-1} (c_j + \epsilon F_j(t)) q^j \tag{2.1}$$

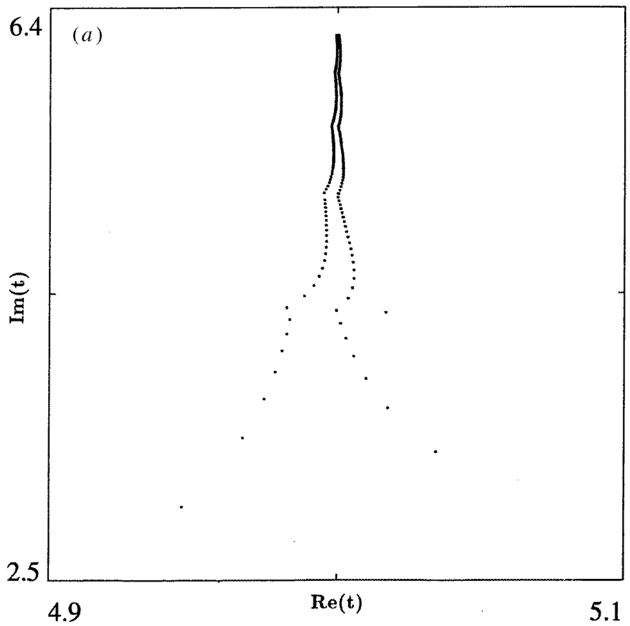
with $\epsilon, c_j \in \mathbb{R}$ and F_j entire functions.

In particular, we look for an approximation of (2.1) around a generic movable critical point t_0, in order to describe locally the properties of the solutions and to construct a model for interpreting theoretically the local singularity structure observed numerically. In order to achieve this we apply the so-called Painlevé α-method assuming that in our asymptotic approximation, which we introduce below, $\alpha = (\log(t - t_0))^{-1/r}$. Numerically (see figures 6 and 7) there is good agreement between the singularities computed by direct integration of (2.1) and those predicted by our local asymptotic approximation.

In appendix A we briefly recall the Painlevé α-method algorithm and show that logarithmic singularities are expected in the solutions of the transformed equations.

Let us consider the case in which the condition of appendix A, $\mathcal{P}_{\bar{k}}(0) \equiv 0$, is not satisfied at the resonance index term; then the local nature of the singularities is logarithmic and the following psi series produces a correct set of conditions for the formal solution of equation (2.1):

$$q(t) = \sum_{k,j \geqslant 0} a_{kj} (t - t_0)^{i(k)+m} [(t - t_0)^r \log (t - t_0)]^j \tag{2.2}$$



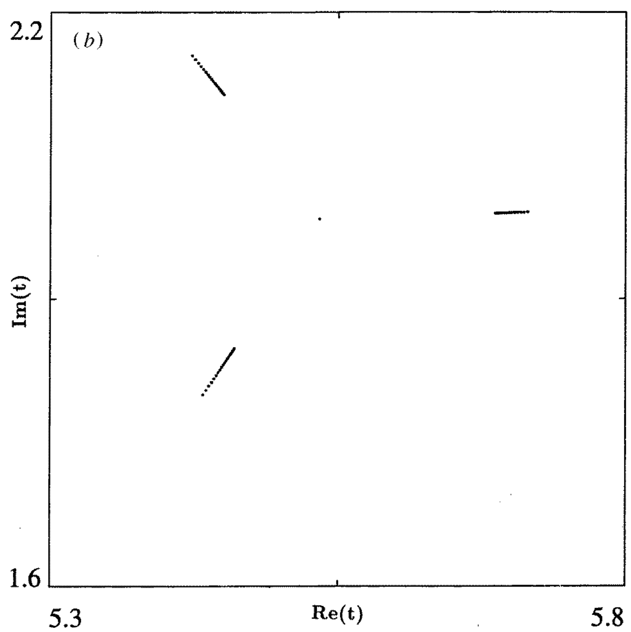


Figure 3. We consider the global (a) and local (b) singularity structure in time of the solution of $\ddot{x} = x(\epsilon \sin(\pi t) - 1) + \epsilon x^2 \sin(t) + x^5$ where $\epsilon = 0.01$ and with initial conditions $x(0) = 0.5$, $\dot{x}(0) = 0$. The singularities in (a) lie on the same Riemann sheet and are obtained by integrating the equation along the real line up to the point $t = 5.0$ and then moving along the imaginary time axis. In (b) we show the local singularity structure associated to the movable critical point $t_0 = (5.5330, 1.9833)$ after projection on the time complex plane. They are obtained by turning 20 times around t_0. In both cases the singularities are identified as algebraic of order $-\frac{1}{2}$ by ATOMFT.

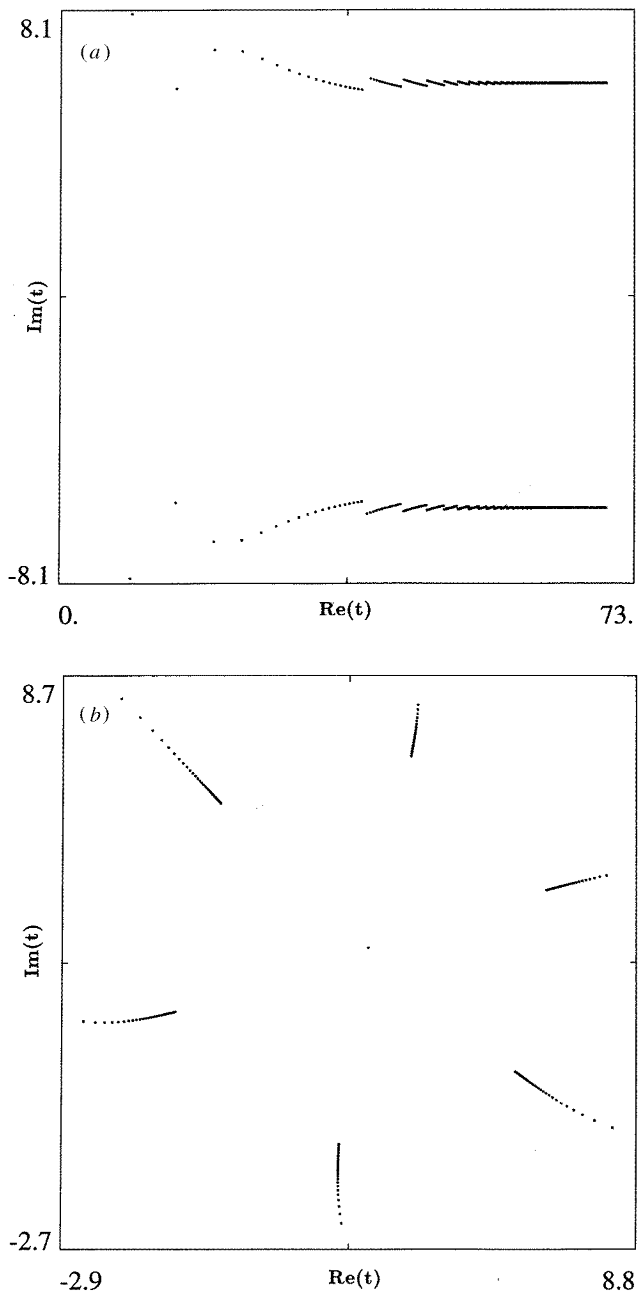


Figure 4. The same as in figure 3 in the case of equation $\ddot{q} = q - (1 + \epsilon_1 \exp(bt) + \epsilon_2 t)q^2$, where $b = 0.3$, $\epsilon_1 = 0.01$, $\epsilon_2 = 2\pi\epsilon_1$, $q(0) = 0.5$, $\dot{q}(0) = 0$ In (a) we first move along the imaginary axis and then parallel to the real one. In (b) $t_0 = (3.3187, 3.2830)$. The singularities are identified as double poles by the ATOMFT program. Notice that in this case the time-dependence is not periodic.

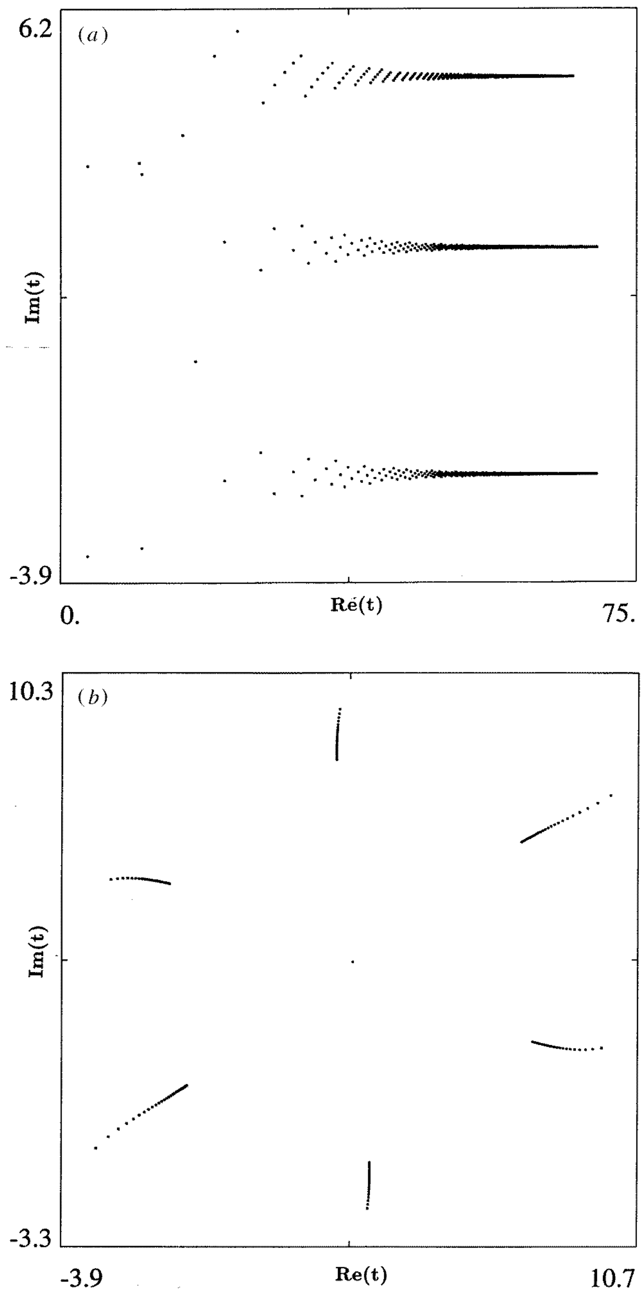


Figure 5. The same as in figure 4 in the case of equation $\ddot{q} = q - (1 + \epsilon \exp(bt))q^2$, where $b = 0.3$, $\epsilon = 0.01$, $q(0) = 0.5$, $\dot{q}(0) = 0$. In (*b*) $t_0 = (3.4503, 3.4562)$. The singularities are identified as double poles by the ATOMFT program. Notice that the potential has imaginary period.

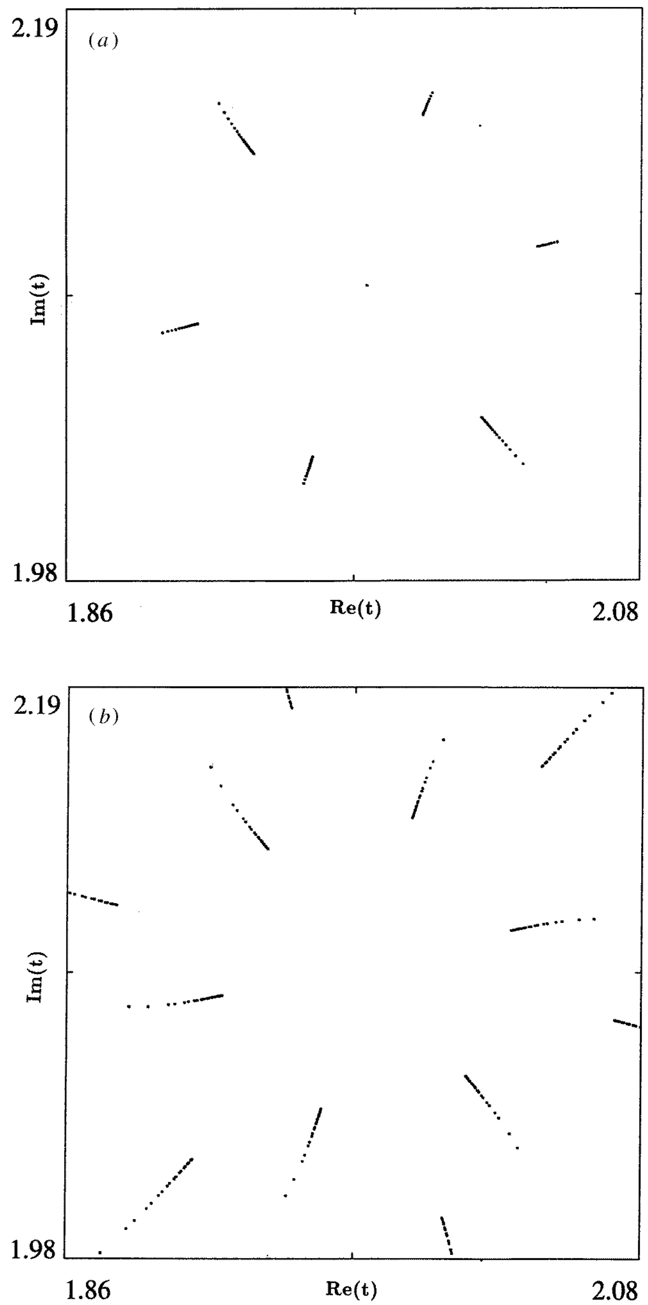


Figure 6. We compare (a) the local singularity structure obtained numerically with ATOMFT as in figure 4 for the point $t_0 = (1.9746, 2.0882)$ relative to equation $\ddot{x} = x(\epsilon \sin(\pi t) - 1) + x^2$, where $\epsilon = 0.01$, $x(0) = 0.5$, $\dot{x}(0) = 0$, with the α-asymptotic perturbation expansion at order 0 (b). The singularities are projected in both cases to the complex plane and are double poles. The conjugated system of order 0 is Painlevé integrable.

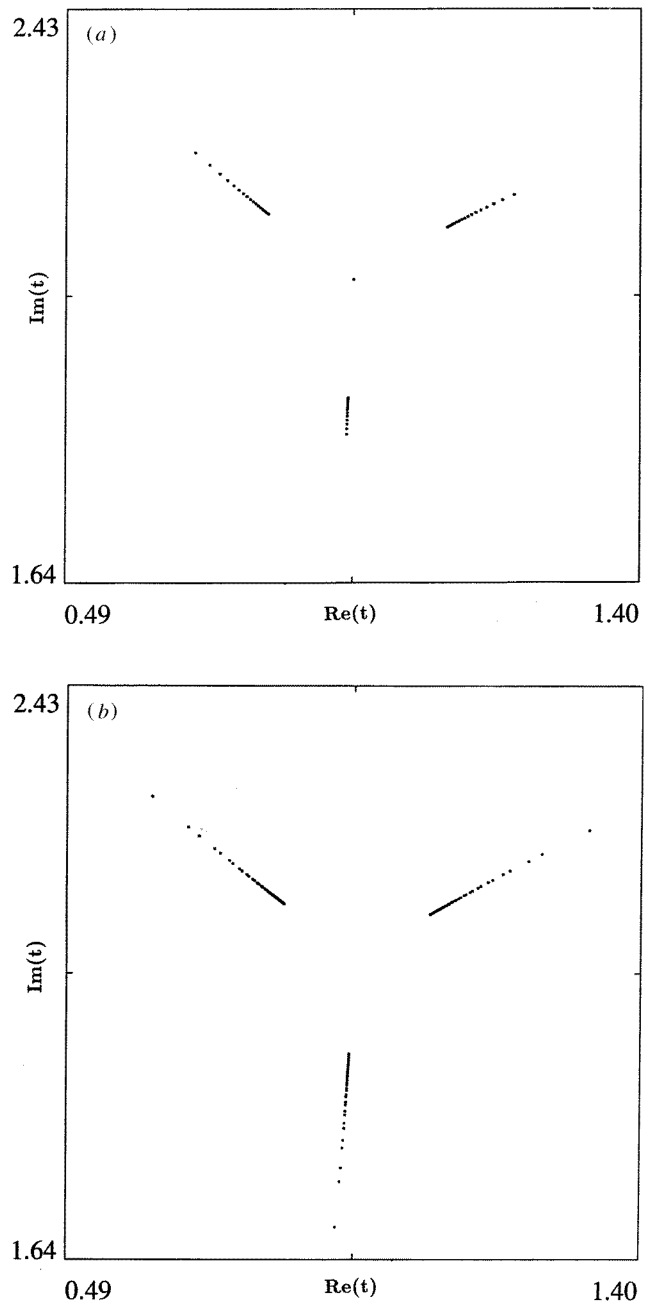
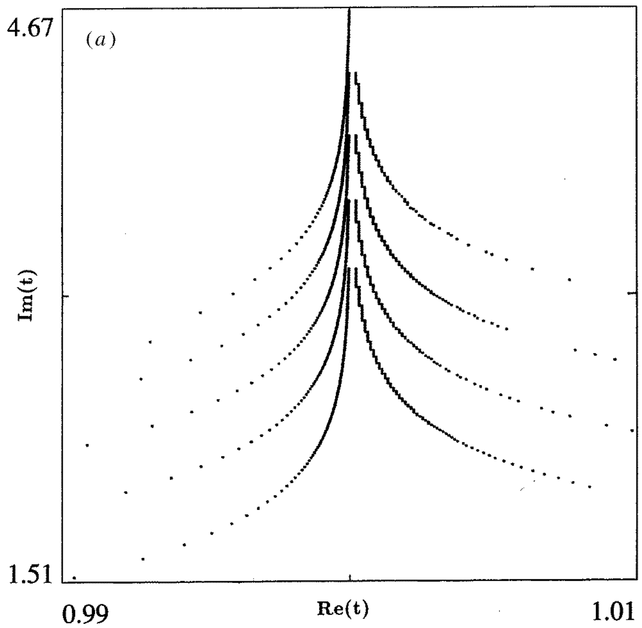


Figure 7. The same as in figure 6, for $t_0 = (0.94504, 2.0563)$ relative to equation $\ddot{x} = x(\epsilon \sin(\pi t) + 1) + x^5$, where $\epsilon = 0.01$, $x(0) = 0.5$, $\dot{x}(0) = 0$, computed with ATOMFT (a) and predicted by the α-asympotic periodic expansion at order 0 (b). The singularities are projected to the complex time plane and are algebraic of order $-1/2$; notice that the conjugated system of order 0 is integrable in the generalized Painlevé sense, since its solutions are square roots of meromorphic functions.



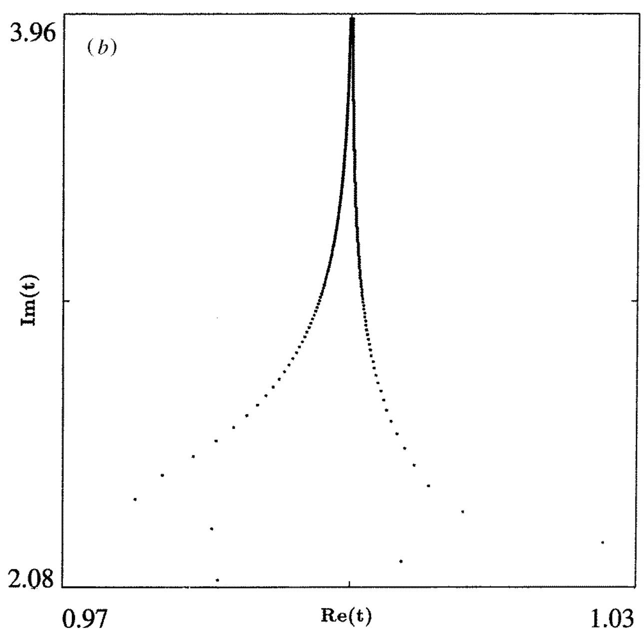
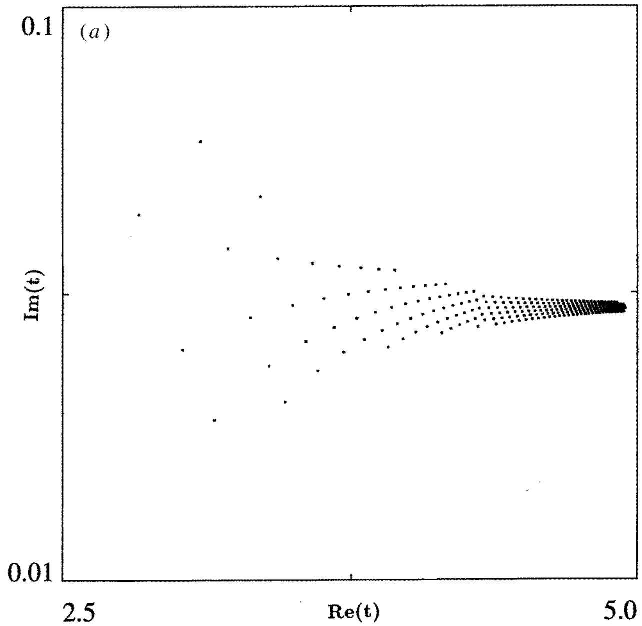


Figure 8. In (a) we superpose the singularity clustering for equation $\ddot{x} = x(\epsilon \sin(\pi t) - 1) + x^2$ where $x(0) = 0.5$, $\dot{x}(0) = 0$, for $\epsilon = 10^{-4}$(= highest 'peak'), 10^{-3}, 10^{-2}, 0.1, 1(= lowest) and integration path along the real time axis up to $t = 1$ and the vertical. The points shown in this picture were used for obtaining table 1. In (b) we plot the complex time singularity clustering for equation $\ddot{x} = (\epsilon \sin(\pi t) - 1)x + x^2$, where $\epsilon = 0.01$, $x(0) = 0.5$, $\dot{x}(0) = 0$ along the path moving along the real time axis up to $t = 1$, then turning around the singularity $t_0 = (0.9861, 2.1008)$ 20 times and then moving up vertically with $\Re(t) = 1$.



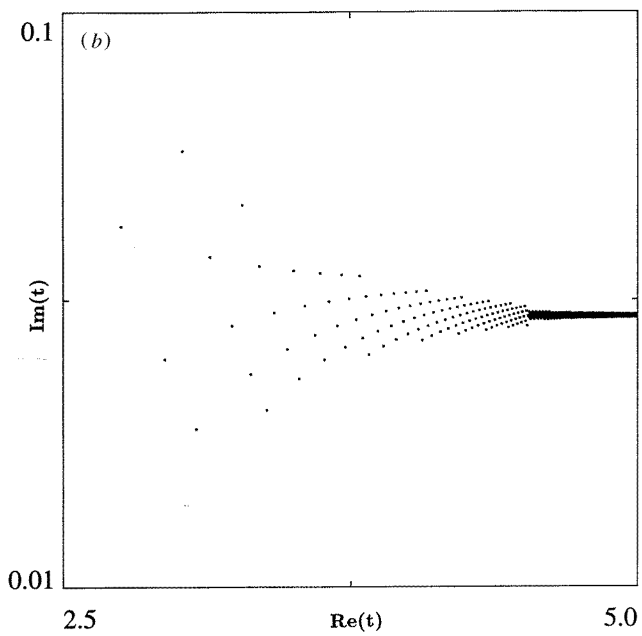


Figure 9. We compare the global singularity structure associated to $\ddot{q} = q - (1 + \epsilon \exp(bt))q^2$ (a) and to $\ddot{q} = q - \epsilon \exp(bt)q^2$ (b), where $\epsilon = 0.01$, $b = 5$, $q(0) = 0.5$, $\dot{q}(0) = 0$. We first integrate the equation along the imaginary axis up to $t_1 = \mathrm{i}\pi/60$ and then parallel to the real axis. Notice that (b) is conjugated to an integrable model using equation (5.2).

where

$$m = -\frac{2}{l-1} \qquad r = 2\frac{l+1}{l-1} \qquad i(k) = \begin{cases} \dfrac{k}{l-1} & \text{if } l \text{ even} \\ \dfrac{2k}{l-1} & \text{if } l \text{ odd.} \end{cases} \tag{2.3}$$

We now consider an asymptotic approximation of $q(t)$ in a neighbourhood of a movable singularity, and relate such approximation to the set of formal solutions obtained using the Painlevé α-method. In particular, we show that the asymptotic equations are exactly those generated by the formal α-method expansion, and so, examining the solutions, we are able to predict the generation of clusters of singularities with logarithmic-type structures, as expected numerically, already at zeroth perturbative order.

Let us denote

$$t - t_0 = \rho \exp(\mathrm{i}\theta).$$

We perform the numerical integration of equation (2.1) in the limit $\rho \to 0$, $\theta \to \infty$. In particular, we may suppose that $\theta \to \infty$ sufficiently fast (in a sense specified below) so that in expansion (2.2) the terms with $k = 0$ are the dominant ones. (In [8] using a different approach the authors consider as dominant the $k = 0$ terms for the Duffing example.)

Let us denote

$$\alpha = [\log(t - t_0)]^{-\frac{1}{r}} \qquad T = \alpha^{-1}(t - t_0) = (t - t_0)\log^{\frac{1}{r}}(t - t_0) \tag{2.4}$$

where r is given by (2.3). We now substitute the Painlevé α-expansion into (2.1), with α and T as above. Then

$$q(t) \approx \alpha^m \sum_{k\geqslant 0} \alpha^{-\frac{i(k)}{r}} q^{(k)}(T) \tag{2.5}$$

where $q^{(k)}$ are the solutions of the Painlevé α-method (see appendix A). We require that, in the limit

$$|T| \to 0 \qquad |T| \gg |t - t_0| \tag{2.6}$$

the terms in (2.5) are dominant. This asymptotic approximation of $q(t)$ is valid when θ goes to infinity sufficiently fast so that, when we insert (2.5) into (2.1), we can consider the terms containing derivatives of α as higher-order corrections at each step. In particular, it is sufficient that

$$\theta > \rho^{\frac{-2r}{r+2}}. \tag{2.7}$$

Notice also that (2.6) and (2.7) form a set of compatible conditions.

Let us now consider the relations between the local formal expansion of $q(t)$ in (2.2) and the one given in (2.5). Let us start from the term $q^{(0)}$, which satisfies the zeroth-order Painlevé equation

$$\frac{\mathrm{d}^2 q^{(0)}}{\mathrm{d}T^2} - (q^{(0)}(T))^l = 0. \tag{2.8}$$

Since T may be chosen to be as small as we like, we may consider the local expansion of $q^{(0)}$ around $T = 0$ and we get

$$q^{(0)}(T) = \sum_{j\geqslant 0} c_j^{(0)} T^{i(j)+m}. \tag{2.9a}$$

If we insert (2.9a) into (2.5) we have that

$$c_j^{(0)} = a_{0,j} \qquad j \geqslant 0 \tag{2.10a}$$

since

$$q(t) \approx (\log(t-t_0))^{\frac{1}{l+1}} q^{(0)}((t-t_0)\log^{\frac{1}{r}}(t-t_0))$$
$$= a_{00}(t-t_0)^m + a_{01}(t-t_0)^{m+r} + \text{h.o.t.}.$$

We have obtained an explicit relation between the coefficients of the local formal expansion (2.2) and the coefficients of the formal solution of (2.8). This last equation admits a first integral which is

$$\frac{1}{2}\frac{\mathrm{d}q^{(0)}}{\mathrm{d}T}^2 - \frac{1}{l+1}(q^{(0)})^{l+1} = m(r+1)a_{00}a_{01}. \tag{2.11}$$

In particular, from (2.10a) and (2.11), we get the following equation:

$$c_r^{(0)} = a_{0,1}. \tag{2.12}$$

This relation means that we are conjugating the local expansion (2.2) around the movable critical point t_0, to the global solution of (2.11), since (2.12) fixes the energy level of (2.11). Then, if we pull back the singularity structure in the T plane associated to $q^{(0)}$, using (2.4), we get an asymptotic approximation of the local structure of $q(t)$ (see figures 6(b) and 7(b)). In section 4, we will analyse an example in order to clarify these ideas.

Let us now consider the terms in (2.5) with $k \geqslant 1$, again in the asymptotic limit (2.6), (2.7). In order to achieve this, we insert the corresponding coefficients in the formal local expansion of $q^{(k)}(T)$, particular solutions of the Painlevé α-equations,

$$q^{(k)}(T) = \sum_{j\geqslant 0} c_j^{(k)} T^{i(j)+m+kr} \qquad k \geqslant 1 \tag{2.9b}$$

in $q(t)$. Then, using the properties of the solutions of the Painlevé α-method, equating the coefficients in (2.2) and (2.7), it is easy to check that

$$c_j^{(k)} = c_{k,j} \qquad j \geqslant 0 \qquad k = 1, \ldots, \bar{k} \qquad \text{where } i(\bar{k}) = r. \tag{2.10b}$$

In fact as soon as $k \geqslant \bar{k}$, new terms containing the logarithm appear (see appendix A) and, in order to compensate them, we should take into account the terms containing derivatives of α which we have neglected so far.

The good agreement between the singularity positions predicted by the asymptotic expansion explained here and those numerically computed starting from (2.1) suggest that (2.5) has to be considered as the first step in the local asymptotic expansion of the solutions of (2.1) in the neighbourhood of a critical singularity. As shown in the figures 6 and 7, the local singularity structure of these examples is highly non-trivial and the complicated local structure is well described by the term $q^{(0)}((t-t_0)\log^{\frac{1}{r}}(t-t_0))$.

3. Convergence of the psi series

In this section (and in appendix B), for simplicity of notations, we consider the case in which equation (2.1) reduces to

$$\ddot{q} = q^l + \sum_{j=0}^{l-1} c_j q^j + \epsilon(F_0(t) + F_1(t)q) \tag{3.1}$$

and we show that the corresponding psi series (see equation (2.2)) actually converges in certain sectors of the complex plane t.

In the more general case of equation (2.2) the estimates we give here are still valid except that we have to estimate more carefully the coefficients in the Taylor expansion of the entire functions $F_j(t)$. Indeed, we estimate from above the coefficients $f_n^{(j)}$ in the Taylor expansions of the entire functions $F_j(t)$, $j = 0, 1$, with an appropriate constant K. Such estimates are not sufficient in order that the proof be self-consistent in the general case of (2.2) and it is sufficient to estimate such terms with Kn^{-d} with $d > 1$.

In any case, our estimates here are not optimal, since our main concern is to show the consistency of the asymptotic approximation proposed in the previous section. Indeed, since the psi series converges, we may resume its terms in the way proposed in the above section. This is of course not a proof of the fact that there are actually secondary singularities on the boundary of the convergence domain, but a first step towards it.

Let us denote

$$\tau = (t - t_0)^{\frac{\beta}{l-1}} \qquad z = \log(t - t_0) \qquad \mu = -\frac{2}{\beta}, \; \bar{r} = \frac{2(l+1)}{\beta} \tag{3.2}$$

where $\beta = 1$ if l is even, $\beta = 2$ elsewhere. We start with the case z real. Then (2.2) becomes

$$q(t) = \sum_{N\geqslant 0} h_N(z)\tau^{N+\mu} \tag{3.3}$$

where

$$h_N(z) = \sum_{k+\bar{r}j=N} a_{kj} z^j \tag{3.4}$$

is a polynomial of degree $[N/\bar{r}]$. Of course

$$\dot{q}(t) = \sum_{N\geqslant 0} g_N(z)\tau^{N+\mu-\frac{(l-1)}{\beta}} \tag{3.5}$$

where

$$g_N(z) = \frac{\mathrm{d}h_N}{\mathrm{d}z} + \frac{\beta(N+\mu)}{(l-1)} h_N(z).$$

When we substitute (3.3) and (3.5) in (2.1), for $N = 0$ we get

$$\frac{\mu\beta}{l-1} h_0 = g_0 \qquad \left(\frac{\mu\beta}{l-1} - 1\right) g_0 = h_0^l$$

so that $h_0 = a_{00}$, $g_0 = -\frac{2}{(l-1)} h_0$, as expected. If $N \geqslant 1$ we get the recursive relations

$$\begin{aligned} &h_N' + \frac{\beta}{l-1}(N+\mu)h_N - g_N = 0 \\ &g_N' + \frac{\beta}{l-1}\left(N + \mu - \frac{l-1}{\beta}\right) g_N - \frac{2l(l+1)}{(l-1)^2} h_N = \tilde{P}_N \end{aligned} \tag{3.6}$$

where

$$\tilde{P}_N = \sum_{\substack{k_1+\cdots+k_l=N \\ k_i\neq N, i=1,\ldots,l}} \prod_{i=1}^{l} h_{k_i} + \sum_{j=0}^{l-1} c_j \sum_{\substack{k_1+\cdots+k_j= \\ N+(l-j)\mu}} \prod_{i=1}^{j} h_{k_i} + f^{(0)}_{\frac{N+l\mu}{l-1}\beta} + \sum_{\substack{\frac{l-1}{\beta}k_0+k_1= \\ N+(l-1)\mu}} f^{(1)}_{k_0} h_{k_1}. \tag{3.7}$$

Equation (3.6) may be rewritten in a compact form as

$$\begin{pmatrix} h_n' \\ g_N' \end{pmatrix} + \mathcal{A}_N \begin{pmatrix} h_N \\ g_N \end{pmatrix} = \begin{pmatrix} 0 \\ \tilde{P}_N \end{pmatrix} \tag{3.8}$$

where

$$\mathcal{A}_N = \begin{pmatrix} \frac{\beta}{l-1}(N+\mu) & -1 \\ \frac{\beta}{l-1}(N+\mu-\frac{l-1}{\beta}) & -\frac{2l(l+1)}{(l-1)^2} \end{pmatrix}. \tag{3.9}$$

Notice that (3.8) becomes equation (A.9) if we consider T as function of z. The diagonal form of $\mathcal{A}_N$ is $\mathcal{D}_N = \mathrm{diag}[\frac{\beta N}{l-1}+1, \frac{\beta N}{l-1}-r]$. The eigenvalues are both positive if $N > \bar{r}$. The matrix which transforms $\mathcal{A}_N$ to $\mathcal{D}_N$ is

$$\mathcal{Q} = \begin{pmatrix} 1 & \frac{l-1}{3l+1} \\ -\frac{l+1}{l-1} & \frac{2l}{3l+1} \end{pmatrix} \tag{3.10}$$

so that

$$\begin{aligned} \begin{pmatrix} h_n \\ g_N \end{pmatrix} &= \int_{-\infty}^{z} \exp\{\mathcal{A}_N(\xi - z)\} \begin{pmatrix} 0 \\ \tilde{P}_N(\xi) \end{pmatrix} \mathrm{d}\xi \\ &= \int_{-\infty}^{z} \mathcal{Q}\exp\{\mathcal{D}_N(\xi - z)\}\mathcal{Q}^{-1} \begin{pmatrix} 0 \\ \tilde{P}_N(\xi) \end{pmatrix} \mathrm{d}\xi \qquad N > \bar{r}. \end{aligned} \tag{3.11}$$

We now need an estimate on the growth of h_N and g_N as functions of N and z. Let us choose the following norm for vectors and matrices:

$$||a_{i,j}|| = \max_{i,j} |a_{ij}|.$$

Then $||\mathcal{Q}|| = ||\mathcal{Q}^{-1}|| = \frac{l+1}{l-1}$, $||\exp\{\mathcal{D}_N(\xi - z)\}|| = \exp\{(\frac{\beta N}{l-1} - r)(\xi - z)\}$ and, using (3.11), we get

$$\max\{|h_N|(z), |g_N|(z)\} \leqslant \left(\frac{l+1}{l-1}\right)^2 \int_{-\infty}^{z} \exp\left\{\left(\frac{\beta N}{l-1} - r\right)(\xi - z)\right\} |\tilde{P}_N(\xi)|\,\mathrm{d}\xi \qquad N > \bar{r},\ z < 0. \tag{3.12}$$

In appendix B we prove that for any $\gamma \in]\gamma^*, 1[$, where $\gamma^* = \frac{l-2}{l-1}$, there exist constants $0 < M < 1$ and $K > 1$ such that

$$\max\{|h_N|(z), |g_N|(z)\} \leqslant M \frac{(K - Kz)^{\frac{N}{\bar{r}}}}{(N+1)^{\gamma}} \qquad \forall N \geqslant 1,\ \forall z < 0. \tag{3.13}$$

Then it easily follows that the series (3.3) converges absolutely for $(K - Kz)^{\frac{1}{\bar{r}}}\tau < 1$ or equivalently for

$$1 - z < \frac{\exp(-rz)}{K}. \tag{3.14}$$

There exists a unique $\bar{z} < 0$ such that equality holds in (3.14) which determines the maximum convergence radius of (2.2).

Let us now consider complex convergence. In analogy with [14], let us define the sets

$$\mathcal{G}_\nu = \{t \in \mathbb{C} : t - t_0 = \rho \mathrm{e}^{\mathrm{i}\theta}, \rho > 0, \nu < \theta < \nu + 2\pi\}. \tag{3.15}$$

If $t \in \mathcal{G}_\nu$, then

$$q(t) = \sum_{N\geqslant 0} \sum_{j=0}^{[N/\bar{r}]} b_j^{(N)} \log^j |t - t_0| \tau^{N+\mu} \tag{3.16}$$

where

$$b_j^{(N)} = \sum_{k=j}^{[N/\bar{r}]} a_{N-\bar{r}k,k} \binom{k}{j} (\mathrm{i}\theta)^{k-j}.$$

Since the form of the series is unaltered, the convergence proof is again the same and it is possible to show absolute convergence in sectors of the τ-complex plane of the form $\tau = |\tau|\mathrm{e}^{\mathrm{i}\phi} = \rho^{\frac{\beta}{l-1}}\mathrm{e}^{\mathrm{i}\phi}$, where $\frac{\nu\beta}{l-1} < \phi < \frac{\nu\beta}{l-1} + \frac{2\pi\beta}{l-1}$.

4. An example

In this section and in appendix C, we give some detailed computations for the following equation:

$$\ddot{q} + q(1 + \epsilon F(t)) - q^2 = 0. \tag{4.1}$$

Notice that the movable singularities for the unperturbed system ($\epsilon = 0$) are double poles.

The same kind of analysis is also possible for the following equation:

$$\ddot{q} + q(1 + \epsilon F(t)) - q^5 = 0 \tag{4.2}$$

for which the movable singularities are algebraic of order $-\frac{1}{2}$. Indeed, in this second case, the corresponding zero Painlevé equation has a general solution which is the square-root of a meromorphic function, according to a theorem by Briot and Bouquet.

Figures 6 and 7, refer to the case where $F(t) = \sin(\Omega t)$, but analogous results may also be shown for more complicated equations like

$$\ddot{q} + q(1 + \epsilon \sin(\pi t)) + \epsilon q^2 \sin(t) - q^5 = 0$$

which do not have periodic perturbations in time (see figure 3(a)). Analyticity is the essential feature of the time-dependent coefficients $F(t)$ in order that the local structure is the one predicted by the computations in the previous sections.

We are going to analyse the behaviour around a movable singularity t_0 and so, in the following, both for (4.1) and (4.2), we substitute F with its Taylor expansion which we suppose convergent in a convenient neighbourhood of a singularity

$$F(t) = \sum_{k=0}^{+\infty} f_k (t - t_0)^k. \tag{4.3}$$

When $\epsilon = 0$, the integral $q(t)$ of (4.1) may be locally represented as a Laurent expansion around the movable time poles

$$q(t) = \sum_{k \geqslant 0} a_k (t - t_0)^{k-2} \tag{4.4}$$

where $a_k \in \mathbb{R}$ satisfy the following recursive relations:

$$\begin{aligned} &[(2k-2)(2k-3) - 2a_0]a_{2k} = -a_{2k-2} + \sum_{j=1}^{k-1} a_{2j} a_{2k-2j} \qquad k \in \mathbb{N} \\ &a_{2k-1} = 0 \end{aligned}$$

initialized by $a_0 = 6$. It is easy to check that the compatibility conditions are satisfied for the resonance term a_6.

If we now try the same type of expansion when $\epsilon \neq 0$, then the dominant behaviour is still $(t - t_0)^{-2}$, but, due to the presence of the perturbative time-dependent term, the compatibility condition for the resonance term a_6 becomes

$$\frac{\epsilon^2 f_1^2}{4} + \epsilon f_2 \left(\frac{1 - f_0 \epsilon}{2} \right) + 6 \epsilon f_4 = 0.$$

Such a condition cannot in general be satisfied by a generic perturbation F, for whatever value of t_0; this means that the Laurent series no longer represents a formal solution to our equation.

If we now substitute the following series:

$$q(t) = \sum_{k,j\geqslant 0} a_{kj}(t-t_0)^{k-2}[(t-t_0)^6 \log(t-t_0)]^j$$
$$= \sum_{k,j\geqslant 0} a_{kj}(t-t_0)^{k-2+6j} \log^j(t-t_0) \tag{4.5}$$

into equation (4.1), we get, for the resonance term a_{60}, the following compatibility equation:

$$7a_{01} = -a_{40} + \epsilon(a_{40}f_0 + a_{30}f_1 + a_{20}f_2 + a_{00}f_4) + 2a_{20}a_{40} + a_{30}^2 \tag{4.6}$$

which assigns the value to a_{01}, because the defining equation of a_{01} is identically satisfied.

Let us now proceed with the α-method (A.2). The only possible choice of the optimal exponent is $m = -2$ and we get the following sequence of differential equations:

$$\begin{aligned}
\mathcal{F}^{(0)} &\equiv \frac{\mathrm{d}^2 q^{(0)}}{\mathrm{d}T^2} - (q^{(0)})^2 = 0\\
\mathcal{F}^{(1)} &\equiv \frac{\mathrm{d}^2 q^{(1)}}{\mathrm{d}T^2} - 2q^{(0)}q^{(1)} = 0\\
\mathcal{F}^{(2)} &\equiv \frac{\mathrm{d}^2 q^{(2)}}{\mathrm{d}T^2} - 2q^{(0)}q^{(2)} - (q^{(1)})^2 + q^{(0)} + \epsilon f_0 q^{(0)} = 0\\
\mathcal{F}^{(3)} &\equiv \frac{\mathrm{d}^2 q^{(3)}}{\mathrm{d}T^2} - 2q^{(0)}q^{(3)} - 2q^{(1)}q^{(2)} + q^{(1)} + \epsilon f_0 q^{(1)} + \epsilon f_1 q^{(0)} T = 0\\
\mathcal{F}^{(k)} &\equiv \frac{\mathrm{d}^2 q^{(k)}}{\mathrm{d}T^2} - 2q^{(0)}q^{(k)} - \mathcal{S}_k(q^{(0)}, \ldots, q^{(k-1)}, T)0 \qquad k \geqslant 4
\end{aligned} \tag{4.7}$$

where, as in (2.4),

$$T = (t-t_0)\log^{\frac{1}{6}}(t-t_0). \tag{4.7a}$$

The first equation has $6\mathcal{P}$ as solution (not on the separatrix), $\mathcal{P}(T-c_0, 0, g_3)$ being the Weierstrass function, while the homogeneous part of the $\mathcal{F}^k$, $k \geqslant 1$ is a Lamé-type equation (see [11]) whose solutions may be written in terms of the Weierstrass function and its derivative, with polar singularities of order $-3, 4$.

As pointed out in [6], the solutions of the homogeneous linear part $\mathcal{F}^{(j)}$, $j \geqslant 1$ do not play any role for what concerns the singularity structure of the solutions, since the singularities generated by them are compensated by singularities of the particular solutions of the inhomogeneous part. So the only singularities appearing in the infinite expansion $q(t)$ produced by the linear equations are those due to the solutions of their inhomogeneous part.

So let us analyse the singularities which show up in the particular solutions of the equations $\mathcal{F}^k$, $k \geqslant 1$. A particular solution may be obtained with the method of variation of constants and, as pointed out in the previous sections, we cannot exclude the presence of movable critical points. In fact, there do necessarily appear movable logarithms in the particular solution to equation

$$\begin{aligned}
\mathcal{F}^{(6)} \equiv \frac{\mathrm{d}^2 q^{(6)}}{\mathrm{d}T^2} &- 2q^{(0)}q^{(6)} + q^{(4)} - 2q^{(1)}q^{(5)} - 2q^{(2)}q^{(4)} - (q^{(3)})^2 + \epsilon f_0 q^{(4)} + \epsilon f_1 q^{(3)} T\\
&+ \epsilon f_2 q^{(2)} T^2 + \epsilon f_3 q^{(1)} T^3 + \epsilon f_4 q^{(0)} T^4 = 0.
\end{aligned}$$

This can be easily checked by considering, for simplicity, the particular solution of $\mathcal{F}^{(0)}$ along the separatrix

$$q^{(0)}(T) = 6(T-c_0)^{-2}.$$

The general solution of the homogeneous linear equation associated to $\mathcal{F}^k$, $k \geqslant 1$ is then

$$q_g^{(k)}(T) = c_1(T - c_0)^4 + c_2(T - c_0)^{-3}$$

and the particular solution of $\mathcal{F}^{(6)}$ obtained with the method of variation of constants is

$$q_p^{(6)}(T) = \frac{(T - c_0)^4}{7}\left[\frac{\epsilon f_2}{2}(1 + \epsilon f_0) + \frac{\epsilon^2 f_1^2}{4} + 6\epsilon f_4\right]\log(T - c_0)) + \text{h.o.t.}. \tag{4.8}$$

So, there does appear a logarithmic singularity with coefficient equal to a_{01} (see (4.6)). In order to remove it, we should consider the trivial case of a constant perturbation. Such logarithmic singularities also appear in the particular solutions outside the separatrix, since (4.8) represents the first-order term of the perturbative expansion of $q_p^{(6)}(T)$ outside the separatrix. (Notice that, contrary to what was claimed in [7] logarithmic singularities also have to appear at the resonance perturbative order for the Duffing example, since their equation falls in the class considered in this paper.)

Let us now consider the limit $(t - t_0) \to 0$ under the condition that the argument of the logarithm is arbitrarily large in absolute value. Under such an assumption, we can neglect all of the terms of the form a_{kj} with $k > 0$, so that the recursive expansion simplifies to

$$(6j - 2)(6j - 3)a_{0j} = \sum_{j_1=0}^{j} a_{0,j_1}a_{0,j-j_1}. \tag{4.9}$$

On the other hand, if we substitute

$$q(t) = q^{(0)}(T)\log^{-\frac{1}{3}}(t - t_0) \tag{4.10}$$

and (4.7a) into (4.1) then, in the asymptotic limit $(t - t_0) \to 0$ with $|(t - t_0)| \ll |T|$, the dominant terms satisfy the following differential equation:

$$\ddot{q}^{(0)}(T) - (q^{(0)})^2(T) = 0. \tag{4.11}$$

Equation (4.11) is just the first equation given by the α-method. The first integral of (4.11) is

$$\tfrac{1}{2}(\dot{q})^2 - \tfrac{1}{3}q^3 \equiv \mathcal{I}_\infty = -84a_{01}. \tag{4.12}$$

This means that we are able to conjugate the local singularity structure of the $q(t)$ solution of (4.1) with the global singularity structure of the $q(T)$ solution of (4.11).

In figure 6(b) we show the pull-back of the singularity nearest to $T = 0$ in the lattice associated to (4.12), inverting (4.7a). Since the argument of $(t - t_0)$ is determined up to integer multiples of 2π, it is clear from (4.7a) that we expect the appearance of $r = 6$ branches of singularities in the t plane as the effect of the pull-back. As appears from figure 6, the agreement between numerical integration of (4.1) and asymptotic approximation explained above is quite good. In fact the structure of singularities of the integrated equation (4.1) appears to depend on the one produced by pulling back the nearest singularity to $T = 0$ in the lattice associated to (4.12) using transformation (4.10) between t and T. Moreover, the same kind of singularity structure also appears in the secondary branches.

5. A model for the global singularity structure

In this section we consider a simple model in order to justify the appearance of chimneys in terms of a certain asymptotic approximation specified below. We also present in this section some numerical analysis also on the global structure of the peaks.

We have observed the appearance of such peaks in many examples in which we have singled out as a common feature the exponential growth of the potential in the time direction in which such structures are detected. Of course the peculiar shape of the peak depends on the particular example considered and at present we do not have a general model in order to explain their appearance and structure. We think in any case that the particular model we present here sheds some light on this subject.

Let us consider the simplest possible model in which one sees the appearance of such chimneys,

$$\ddot{q} = q - \epsilon \exp(bt) q^2 \tag{5.1}$$

where we take $b > 0$ and $\epsilon \neq 0$ (fixed constant). Then we apply the following transformation:

$$q(t) = \exp(-t)\Theta(z) \qquad z = \exp(2t) \tag{5.2}$$

to (5.1). We then get the following equation for Θ:

$$\ddot{\Theta} = -\tfrac{1}{4}\epsilon z^{\frac{(b-5)}{2}} \Theta^2. \tag{5.3}$$

When $b = 5$, the asymptotic conjugated model is an integrable one in the Painlevé sense. The corresponding equation is in fact

$$\ddot{\Theta} + \frac{\epsilon}{4}\Theta^2(z) = 0$$

whose general solution $\Theta(z) = 6\mathcal{P}(\mathrm{i}\frac{\sqrt{\epsilon}}{2} z - c_0, 0, g_3)$ is the Weierstrass function. The singularities of Θ are double poles disposed on a regular lattice with two fundamental periods $2\omega_1, 2\omega_2$. This lattice, through the transformation $z = \exp(2t)$, is pulled back into a chimney of singularities which we show in figure 9(b). Then it is clear that the singularities in t of q are logarithmic in nature due to (5.2); moreover, the solution in this case is explicit and is given by

$$q(t) = 6\exp(-t)\mathcal{P}\left(\mathrm{i}\frac{\sqrt{\epsilon}}{2}\exp(2t) - c_0, 0, g_3\right).$$

So the lattice of singularities of $\mathcal{P}(z)$ is squeezed towards the t integration path by the above transformation at exponential rate.

In the case $b = 5$, we may then also give a bound on the rate of approximation of the singularities pulled back from (5.3) to the integration path. Indeed, let us consider, for simplicity, the case in which the lattice of singularities associated to (5.3) is rectangular, and denote the basic periods with $(2\omega_1, 2\mathrm{i}\omega_2)$, with $\omega_1, \omega_2 \in \mathbb{R}$. Choose a path along the z-real axis. Then the distance between the path and the singularities of the lattice, $z_n = 2n\omega_1 + \omega_2$, which are nearest to the path is constant and equal to

$$\inf_{z\in\mathbb{R}} |z - z_n| = |\omega_2|.$$

If we pull back the lattice of singularities using the transformation

$$z = \exp(2t) \qquad z_n = \exp(2t_n)$$

we easily get that the distance between the corresponding points in the time plane is given by

$$\inf_{t\in\mathbb{R}} |t - t_n| = \frac{1}{n}\left|\frac{\omega_2}{4\omega_1}\right|(1 + o(1)) \tag{5.4}$$

where $o(1) \to 0$ as $n \to \infty$. That is, the singularities of the model approach the integration path at a rate which allows for infinite real-time analytic continuation of the solutions.

Moreover, if one considers a slightly more general model

$$\ddot{q} = q - (1 + \epsilon \exp(bt))q^2 \tag{5.5}$$

with $b > 0, \epsilon \neq 0$ as before, then we may take (5.1) as its asymptotic approximation for large real times (compare figures 9(a) and (b)).

We now briefly describe some of the numerical analyses we made on model (4.1) and in particular on the following equation:

$$\ddot{q} = -q(1 + \epsilon \sin(\pi t)) + q^2. \tag{5.6}$$

The peaks are obtained by integrating the equations starting from the time origin along a path which lies in the real time axis up to the point $(t_0, 0)$ and then moving along increasing or decreasing imaginary times for $\Re t = t_0$. Then, whatever the initial conditions, the degree of $\mathcal{V}, \mathcal{R}$ and $\epsilon \neq 0$ are, more and more singularities appear which look as if they approach the integration path (see figures 1, 2, 3(a), 4(a), 5(a) and 9). Such chimneys also appear if we make a certain number of turns around some local singularity and then move up (see figure 8(b)) and seem to appear independently of the choice of t_0 whatever the value of ϵ (see figure 8(a)).

In particular, we tried to check whether the ratio of approximation of singularities to the integration path for model (5.6) had the same rate as that of model (5.1). Indeed, we are convinced that such rate should depend only on l. The major problem is that we cannot control whether the ATOMFT program detects all interesting movable singularities along a certain path. In this respect, a comparison with other numerical methods for detecting singularities would be extremely interesting.

Under the assumption that we do not lose 'too many' singularities with ATOMFT, tables 1 and 2 show that there is a scaling invariance as we change ϵ keeping all other quantities fixed. The 'peak' moves without changing its form towards the real time axis as ϵ increases. The right (+) and the left (−) branches approach the integration path at a negative rate, approximately proportional to n^{-1} (see the first column in tables 1 and 2) in rough agreement with (5.4), for example (5.1).

Table 1. We plot the interpolated angular coefficients of the singularities $t^{(n)} = (t_R^{(n)}, t_I^{(n)})$ to the left of the integration path versus $\log n$, the order of appearance of $t^{(n)}$ for various values of ϵ and for the same initial conditions $x(0) = 0.5$, $\dot{x}(0) = 0$ and the same time integration path (along the real axis up to $t = 1$, and then parallel to the imaginary time axis). The last line refers to the case of figure 9, in which we make 20 turns around the first singularity along the chimney before continuing in the imaginary direction.

	Interpolated angular coefficients for left approximation				
ϵ	$\log(t_R^{(n)} - t_0)$	$\log(t_I^{(n)})$	$\log(t_R^{(n+1)} - t_R^{(n)})$	$\log(t_I^{(n+1)} - t_I^{(n)})$	$\log(t^{(n)} - t^{(n)})$
10^{-4}	−0.95	0.08	−1.86	−0.96	−0.96
10^{-3}	−0.96	0.09	−1.88	−0.96	−0.96
10^{-2}	−0.96	0.10	−1.89	−0.96	−0.96
0.1	−0.96	0.11	−1.91	−0.95	−0.95
1.0	−0.96	0.13	−1.91	−0.95	−0.95
160.0	−0.999	0.21	−1.97	−0.99	−0.99
0.01*	−0.90	0.09	−1.78	−0.90	−0.90

Table 2. We plot the interpolated angular coefficients of the singularities $t^{(n)} = (t_R^{(n)}, t_I^{(n)})$ to the right of the integration path versus $\log n$, the order of appearance of $t^{(n)}$ for various values of ϵ and for the same initial conditions $x(0) = 0.5$, $\dot{x}(0) = 0$ and the same time integration path (along the real axis up to $t = 1$, and then parallel to the imaginary time axis). The last line refers to the case of figure 9, in which we make 20 turns around the first singularity along the chimney before continuing in the imaginary direction.

	Interpolated angular coefficients for right approximation				
ϵ	$\log(t_R^{(n)} - t_0)$	$\log(t_I^{(n)})$	$\log(t_R^{(n+1)} - t_R^{(n)})$	$\log(t_I^{(n+1)} - t_I^{(n)})$	$\log(t^{(n)} - t^{(n)})$
10^{-8}	−0.99	0.06	−2.02	−0.93	−0.93
10^{-3}	−0.81	0.07	−1.62	−0.81	−0.81
10^{-2}	−0.96	0.08	−1.64	−0.84	−0.84
0.1	−0.99	0.11	−1.87	−0.92	−0.92
1.0	−0.86	0.11	−1.60	−0.85	−0.85
160.0	−0.987	0.21	−2.05	−0.97	−0.97
0.01*	−0.96	0.10	−1.72	−0.93	−0.93

6. Conclusions

The local asymptotic analysis presented in this paper shows a commonly expected behaviour of the singularities of all Hamiltonian systems perturbed with time-analytic perturbations in class (1.1). In particular, we have proposed an asymptotic expansion of the solutions of (1.1) modelled on the Painlevé α-method where we take the parameter α itself as a function of time.

In this way, we can get an asymptotic conjugation between the local singularity structure of the original model (1.1) and the global singularity structure of the zeroth Painlevé equation using transformation (2.4). Notice that our model contains as a subcase the results presented in [8].

Numerically there is good agreement between the theoretical predictions obtained with perturbative expansions and asymptotic approximations, in the cases where the resonance r is an integer number.

The cases in which r is rational are even more delicate. In fact, the global structure of the conjugated system (2.8) is no longer integrable in the sense that even at zeroth order, the solution $q^{(0)}$ is not uniformizable. That is, the conjugated system already exhibits an infinite-sheeted Riemann structure. In order to see numerically the local 'branches' of singularities, we have to move at a small but finite distance from the singularity. So we think that numerical problems may show up due to the extreme difficulty of controlling the way in which we change the sheet, integrating the equation along complicated paths. We think that more complicated cases such as these deserve study in themselves.

In any case, we still lack an analytic proof of the convergence of the method and of the fact that secondary singularities should appear and form the structures shown in this paper. In section 3, as a first step towards this goal we have shown that the psi series converges absolutely. Unfortunately the convergence proof presented here and analogous proofs considered in other classes of examples (see [14]) do not allow for explicit estimates of the radius of convergence.

In any case, we are convinced that the singularities we detect numerically are on the boundary of the domain of convergence of the psi series. We also believe that the α-method gives the optimal way in which one should resum the series in order to prove the appearance of secondary singularities. Of course psi series are extremely complicated objects and it is not known in general that singularities appear on the boundary of the convergence domain (see [11] for instance).

In section 5 we have considered the global singularity structure associated to the solutions of (1.1) in the case in which there is exponential growth in time of the potential in some directions. We conjecture that in this case, one has to expect the appearance of ‘barriers’ of singularities integrating (1.1) in directions parallel to the exponential growth of the potential. We got numerical evidence of this conjecture in all the examples considered.

We have proposed here a simple basic example which may be considered as the simplest model for further investigation on the nature and structure of these ‘chimneys’. The model considered here allows for infinite-time analytic continuation of the solution since the singularities approach the path of integration at sufficiently low speed. We are going to investigate this and other models in order to construct an asymptotic approximation for large times. We are convinced that the structure of the ‘chimneys’ is strictly related to the asymptotic properties in the time of the potential, more than with the integrability or non-integrability of the model.

Preliminary numerical investigations on a restricted class of models show that the form of the peaks depends on the system under consideration, and depends very little on the value of the perturbation parameter ϵ (in a certain range of ϵ). The peaks look self-similar and, as ϵ grows, shift towards the real axis without changing of form.

When $\deg \mathcal{V} = 2$, the singularities move towards the integration path at roughly n^{-1} speed, where n is the order of appearance of the singularity in the chimney. Such results are in agreement with what is evidenced by our simple model.

Finally, the methods explained here about the local analysis of the time singularities may be applied in principle also to a more general class of Hamiltonian systems in which $\mathcal{H}$ is algebraic in p, rational in q and analytic in t, since Painlevé α-method still works.

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Appendix A

Let us rewrite (2.1) as

$$\mathcal{F}(q, t) = 0. \tag{A.1}$$

To (A.1), we may formally associate an infinite sequence of equations obtained by using the following algorithm originally introduced by Painlevé in order to get the necessary conditions for Painlevé integrability (see [14] and [11]).

Let $t_0 \in \mathbb{C}$ (in our class of examples there are no fixed singularities), α be a non-zero complex parameter, and let us consider the following perturbation:

$$\alpha \neq 0 : t = t_0 + \alpha T \qquad q = \alpha^m \sum_{k=0}^{+\infty} \alpha^{i(k)} q^{(k)} : \mathcal{F} = \alpha^s \sum_{k=0}^{+\infty} \alpha^{i(k)} \mathcal{F}^{(k)} \tag{A.2}$$

where m is rational and has to be chosen optimally; s is constant and is totally determined by m; $i(k)$ are sequences of rational numbers determined by imposing compatibility conditions in the recursion relations in (A.2).

Then, at perturbation order zero, all coefficients of equation $\mathcal{F}^{(0)}$ are constant and, for a suitable choice of m, only a few terms survive. Following [7], we call a simplified equation, the equation of order zero associated to a given perturbation

$$\mathcal{F}^{(0)}(q^{(0)}) = 0. \tag{A.3}$$

The successive steps of the α-method consist of:

(1) determining all sequences m such that the perturbation (A.2) verifies (A.3);

(2) finding the general solution of the simplified equation;

(3) defining, for each $k \geqslant 1$, $q^{(k)}$ as a particular solution of equation $\mathcal{F}^{(k)} = 0$.

In our setting, $\mathcal{F}^{(k)}, k \geqslant 1$ is linear with the same homogeneous non-homogeneous parts depending on the previous terms $q^{(0)}, \dots, q^{(k-1)}$ and on T. Of course in the Painlevé integrable cases—which are not present in our setting—we have that $q^{(k)}$ is free from movable critical points, in order to satisfy stability for all sequences of perturbed equations.

It is easy to realize that the right choice of m is nothing other than the rational power corresponding to the dominant behaviour of the movable singular points, that is

$$m = -\frac{2}{l-1} \tag{A.4}$$

while the indices $i(k) = \frac{\beta k}{l-1}$ are as in (2.3). Moreover, the dominant part of the solutions of $q^{(k)}(T)$ around the movable singularities is $a_{k,0}T^{i(k)+m}$ up to order k which corresponds to the so-called resonance index. In our case, the resonance is

$$r = 2\frac{l+1}{l-1}. \tag{A.5}$$

We now prove that the dominant term of the corresponding particular solution to equation $\mathcal{F}^{(r)}$ presents a logarithmic singularity of type $T^{r+m}\log(T)$, unless certain very peculiar conditions in the dependence on time of $R(t,q)$ are satisfied. These conditions are, of course, exactly the same as those which have to be satisfied in (2.3) in order that $a_{k,j} = 0$ if $j \neq 0$.

For simplicity, we check the appearance of logarithmic singularities for the first time at the index $i(\bar{k}) = r$ when (a.2) is initialized by the 'separatrix' solution $q^{(0)}(T) = c(T - t_1)^m$, of $\mathcal{F}^{(0)} = \ddot{q}^{(0)} - Aq^{(0)l} = 0$, where $c^{l-1} = \frac{m(m-1)}{A}$ and t_1 depends on the initial condition. Indeed, if we prove the appearance of logarithmic time singularities at index $i(\bar{k}) = r$ for the first time in this particular case, then logarithmic singularities have to appear, for the first time, at the same recursive order r, also in the generic one since, locally, the dominant behaviour of such generic solutions near the singularities is just the exact behaviour of the 'separatrix' solution. Let

$$\ddot{q} = q^l + \mathcal{W}(q, t, \epsilon) \tag{A.6}$$

where $\mathcal{W}$ is algebraic of degree at most $l-1$ in q and analytic in t. Then $\mathcal{F}^{(0)} = \ddot{q}^{(0)} - q^{(0)^l} = 0$ has separatrix solution

$$q^{(0)}(T) = \left[\frac{2l+2}{(l-1)^2}\right]^{\frac{1}{l-1}} (T - t_1)^{-\frac{2}{l-1}} \tag{A.7}$$

equations $\mathcal{F}^{(k)} = 0, k \geqslant 1$, are all linear with homogeneous part

$$\ddot{w} - l(q^{(0)})^{l-1}w = 0 \tag{A.8}$$

and have basis of solutions given by $w_1(T) = (T - t_1)^{-\frac{l+1}{l-1}}$ and $w_2(T) = (T - t_1)^{\frac{2}{l-1}}$. The complete equation at order $k \geqslant 1$ is

$$\ddot{q}^{(k)} - l(q^{(0)})^{l-1}q^{(k)} = \mathcal{P}_k(q^{(0)}, \ldots, q^{(k-1)}, T, \epsilon) \tag{A.9}$$

with $\mathcal{P}_k$ a homogeneous polynomial of degree $m + i(k) - 2$ in the $q^{(j)}$ and in $f_j^{(i)}$, the coefficients of the Taylor expansion of the time-dependent coefficients of $\mathcal{F}_i$ in (2.2).

The particular solutions $q_k^P(T)$ can be obtained using the method of variation of constants

$$q_k^P(T) = -q_k^{(1)}(T)\int^T \frac{q_k^{(2)}\mathcal{P}_k}{\Delta} + q_k^{(2)}(T)\int^T \frac{q_k^{(1)}\mathcal{P}_k}{\Delta}$$

where $q_k^{(1)}$ and $q_k^{(2)}$ are the two independent solutions of the corresponding homogeneous equation listed above and

$$\Delta(T) = q_k^{(1)}(T)\dot{q}_k^{(2)}(T) - q_k^{(2)}(T)\dot{q}_k^{(1)}(T) = (3l+1)/(l-1)q^{(1)}(0)q^{(2)}(0)$$

is the Wronskian.

For $k < \bar{k}$, $q_P^{(k)}(T) = (T-t_1)^{\nu_k^{(1)}}\int^T (T-t_1)^{\mu_k^{(1)}} + (T-t_1)^{\nu_k^{(2)}}\int^T (T-t_1)^{\mu_k^{(2)}}$ with $\mu_k^{(1)} < -1$ and $\mu_k^{(2)} > 0$ and for $k = \bar{k}$,

$$q_{\bar{k}}^P(T) = \frac{l-1}{3l+1}q_{\bar{k}}^{(1)}(0)q_{\bar{k}}^{(2)}(0) \times \tilde{\mathcal{P}}_{\bar{k}}(0)\left[-(T-t_1)^{\frac{2l}{l-1}}\int^T (T-t_1)^{-1} + (T-t_1)^{-\frac{l+1}{l-1}}\int^T (T-t_1)^{2\frac{l+1}{l-1}}\right].$$

Then logarithmic singularities are absent if and only if $\tilde{\mathcal{P}}_{\bar{k}}(0) \equiv 0$. This ends the proof since this last condition cannot be satisfied by a generic non-trivial time-analytic perturbation $\mathcal{R}$ independently of t_0. Notice that $\mathcal{P}_k \equiv \tilde{\mathcal{P}}_{\bar{k}}(0)(T-t_1)^{m+i(k)-2}$ in the separatrix case.

Appendix B

Using the notations of section 3, below we show by induction that the following proposition is true.

Proposition B.1. For all $l \geqslant 2$, let us define

$$\gamma^* = \frac{l-2}{l-1}.$$

Then, $\forall\gamma \in]\gamma^*, 1[$ it is possible to find constants $0 < M < 1$ and $K \geqslant 0$ such that

$$\max\{|h_N(z)|, |g_N(z)|\} \leqslant M\frac{(K - Kz)^{\frac{N}{r}\gamma}}{(N+1)} \qquad \forall N \geqslant 1,\ \forall z < 0 \tag{B.1}$$

where $h_N(z)$ and $g_N(z)$ are given by (3.8).

The case $l = 3$ and $F_1(t) \equiv 0$ (Duffing equation) has been considered in [14]. Notice that the optimal exponent in (B.1) is γ^*. Indeed, in that case $K \equiv K(\gamma, M)$ is the smallest possible. We believe that it is possible, in the generic case, to achieve γ^*, but this requires more delicate estimates than those presented here. Actually in [14], the authors have proved that in general it is possible to take $\gamma = \frac{1}{2}$, if $l = 3$ and $F_1 \equiv 0$. In any case neither proof produces explicit estimates of K.

Proof of proposition B.1. In analogy with [14], we divide the proof into two parts. First, from lemma B.2, we directly estimate the first $\bar{N}$ terms with (B.1), if M and K are chosen conveniently. The second part is by induction on N, and fixes γ^*.

Lemma B.2. Let $\bar{r} \geqslant 1, \gamma > 0$ be fixed and let $\mathcal{K}_N(z) = \sum_{i=0}^{N/\bar{r}} c_i^{(N)} z^i$ be any sequence of polynomials with $N \geqslant 1$. Then, for all $\bar{N} > \bar{r}$ and for all $0 < M < 1$ there exists $K > 1$ such that

$$|\mathcal{K}_N(z)\| \leqslant \frac{M}{(N+1)^\gamma}(K - Kz)^{\frac{N}{\bar{r}}} \qquad \forall z < 0, \ N = 1, \ldots, \bar{N}.$$

Lemma B.2 is analogous to lemma 4.3 in [14] and its proof is trivial. We may apply lemma B.2 to $\mathcal{K}_N = h_N, g_N$ so that (B.1) is true for $N = 1, \ldots, \bar{N}$. We now proceed by induction starting with an estimate on $\tilde{P}_N$. The following inequality easily follows from (3.7) using estimate (b.1) true for $n < N$:

$$\begin{aligned}|\tilde{P}_N(z)| \leqslant{}& M^l (K - Kz)^{\frac{N}{\bar{r}}} \sum_{\substack{k_1+\cdots+k_l=N \\ k_i \neq N, i=1,\ldots,l}} \prod_{i=1}^{l} (k_i+1)^{-\gamma} \\ &+ \sum_{j=0}^{l-1} c_j M^j (K-Kz)^{\frac{N+(l-j)\mu}{\bar{r}}} \sum_{\substack{k_1+\cdots+k_j= \\ N+(l-j)\mu}} \prod_{i=1}^{j} (k_i+1)^{-\gamma} + f^{(0)}_{\frac{N+l\mu}{l-1}\beta} \\ &+ M \sum_{k_0=0}^{\frac{\beta(N+(l-1)\mu)}{l-1}} f^{(1)}_{k_0} (K-Kz)^{\frac{N+(l-1)\mu-\frac{l-1}{\beta}k_0}{\bar{r}}} \left(N + (l-1)\mu - \frac{l-1}{\beta}k_0 + 1\right)^{-\gamma}.\end{aligned} \tag{B.2}$$

First, since $F^{(j)}(t)$, $j = 0, 1$ are entire functions, we may estimate $f_k^{(j)} \leqslant K$, $\forall k \geqslant 0$, $j = 0, 1$. Moreover, it is not restrictive to suppose $c_j \leqslant K$, $j = 0, \ldots, l-1$. The major concern is in estimating sums of the form

$$A(N, j) = \sum_{k_1+\cdots+k_j=N} \prod_{i=1}^{j} (k_i+1)^{-\gamma} \qquad j = 1, \ldots, l, \ N \geqslant 1. \tag{B.3}$$

If $j = 0$ the sum reduces to a constant term when $N = -l\mu$ and is zero otherwise. If $j = 1$, $A(N, 1) = (N+1)^{-\gamma}$. If $j = 2$ and $\gamma < 1$, we easily get that

$$\begin{aligned}A(N, 2) &= 2 \sum_{k_1=0}^{\frac{N}{2}} (k_1+1)^{-\gamma} (N+1-k_1)^{-\gamma} \\ &\leqslant 2\left(\frac{N}{2}+1\right)^{1-2\gamma} \int_0^1 \mathrm{d}x (1-x^2)^{-\gamma} \leqslant K(N+2)^{1-2\gamma}.\end{aligned} \tag{B.4}$$

In general, if $j > 2$, we have that

$$A(N, j) \leqslant A(N, 2)\left(\sum_{k=0}^{N} (k+1)^{-\gamma}\right)^{j-2} \leqslant \frac{K(N+2)^{1-2\gamma}(N+1)^{(1-\gamma)(j-2)}}{(1-\gamma)^{j-2}}. \tag{B.5}$$

We are now able to estimate (B.2), using (B.3), (B.4) and (B.5):

$$\begin{aligned}|\tilde{P}_N| \leqslant{}& \frac{M^l K}{(1-\gamma)^{l-2}} (N+2)^{1-2\gamma} (N+1)^{(1-\gamma)(l-2)} (K-Kz)^{\frac{N}{\bar{r}}} \\ &+ K\delta_{N+l\mu} + K(N+1)^{-\gamma}(K-Kz)^{\frac{N+(l-1)\mu}{\bar{r}}} \\ &+ K^2 (N+2)^{1-2\gamma} \sum_{j=2}^{l-1} \frac{M^j (N+1)^{(1-\gamma)(j-2)}}{(1-\gamma)^{l-3}} (K-Kz)^{\frac{N+(l-j)\mu}{\bar{r}}}\end{aligned}$$

$$+K + KM(N+1)^{-\gamma}\sum_{k_0=0}^{N}(K-Kz)^{\frac{N+(l-1)\mu}{\bar r}-\frac{(l-1)k_0}{\beta\bar r}}$$

$$\leqslant \frac{(l+2)M^l K}{(1-\gamma)^{l-2}}(N+2)^{1-2\gamma}(N+1)^{(1-\gamma)(l-2)}(K-Kz)^{\frac{N}{\bar r}}$$

$$+KM(N+1)^{-\gamma+1}(K-Kz)^{\frac{N+(l-1)\mu}{\bar r}}. \tag{B.6}$$

We now use the following technical lemma (see also [14]).

Lemma B.3. Let $K > 0$. Then, $\forall N > \bar r$ and $\forall z < 0$ the following inequality holds:

$$I(z) \equiv \int_{-\infty}^{z} \exp\left\{\left(\frac{\beta N}{l-1} - r\right)(\xi - \zeta)\right\}(K-K\xi)^{\frac{N}{\bar r}}\,\mathrm{d}\xi \leqslant \frac{(K-Kz)^{\frac{N}{\bar r}}}{(N-\bar r)}\delta_l \tag{B.7}$$

where $\delta_l = (l-1)\frac{2(l^2-1)+\beta(l+2)}{\beta(l+3)}$.

Proof of lemma B.3. Using integration by parts, as in [13], we get

$$I(z) = (K-Kz)^{\frac{N}{\bar r}}\left(\frac{\beta N}{l-1} - r\right)^{-1} + \frac{NK}{\bar r}(K-Kz)^{\frac{N}{\bar r}-1}\left(\frac{\beta N}{l-1} - r\right)^{-2} + \cdots$$

$$+\frac{N}{\bar r}\left(\frac{N}{\bar r}-1\right)\cdots\left(\frac{N}{\bar r}-\left[\frac{N}{\bar r}\right]+1\right)K^N(K-Kz)^{\frac{N}{\bar r}-[\frac{N}{\bar r}]}\left(\frac{\beta N}{l-1} - r\right)^{-[\frac{N}{\bar r}]1}$$

$$+\frac{N}{\bar r}\left(\frac{N}{\bar r}-1\right)\cdots\left(\frac{N}{\bar r}-\left[\frac{N}{\bar r}\right]\right)K^{N+1}\left(\frac{\beta N}{l-1} - r\right)^{-[\frac{N}{\bar r}]-1}$$

$$\times\int_{-\infty}^{z}\exp\left\{\left(\frac{\beta N}{l-1} - r\right)(\xi - z)\right\}(K-K\xi)^{\frac{N}{\bar r}-[\frac{N}{\bar r}]-1}. \tag{B.8}$$

Since $\xi \in]-\infty, z[$, we get

$$\int_{-\infty}^{z}\exp\left\{\left(\frac{\beta N}{l-1} - r\right)(\xi - z)\right\}(K-K\xi)^{\frac{N}{\bar r}-[\frac{N}{\bar r}]-1} \leqslant \frac{(K-Kz)^{\frac{N}{\bar r}-[\frac{N}{\bar r}]-1}}{\left(\frac{\beta N}{l-1}-r\right)}.$$

Then (B.8) becomes

$$I(z) \leqslant (K-Kz)^{\frac{N}{\bar r}}\left(\frac{\beta N}{l-1} - r\right)^{-1}\left\{1 + \frac{N}{\bar r}\left(\frac{\beta N}{l-1} - r\right)^{-1}\right.$$

$$\left.\times(1-z)^{-1}\left[1+\sum_{j=1}^{N}(1-z)^{-j}\left(\frac{\beta N}{l-1} - r\right)^{-j}\prod_{i=1}^{j}\left(\frac{N}{\bar r}-i\right)\right]\right\}$$

$$\leqslant (K-Kz)^{\frac{N}{\bar r}}(\beta N - 2(l+1))^{-1}(l-1)\left\{1+\left(1+\frac{2(l+1)}{\beta}\right)\frac{l-1}{l+3}\right\}$$

from which we immediately get estimate (B.7). We now substitute (B.6) into (3.12) and use (B.7), and we get

$$\max\{h_N(z), g_N(z)\} \leqslant \frac{(l+2)\delta_l M^l K}{(1-\gamma)^{l-2}}(N+1)^{(1-\gamma)(l-2)-2\gamma}(K-Kz)^{\frac{N}{\bar r}}$$

$$+KM\delta_l(N+1)^{-\gamma}(K-Kz)^{\frac{N+(l-1)\mu}{\bar r}}. \tag{B.9}$$

The proof of proposition B.1 is now finished since it is sufficient to choose $\gamma > \frac{l-2}{l-1}$ in (B.9) in order that (B.1) is satisfied for N sufficiently big. Indeed, the second term in (B.9) is dominated by the first one. In order to ensure that for $\gamma = \gamma^*$ our estimates are still valid, it is sufficient that

$$\frac{2(l+2)\delta_l M^{l-1} K}{(1-\gamma^*)^{l-2}} < 1.$$

Appendix C

Here we give some details on the properties of the solutions of the infinite set of equations obtained with the α-method in the example considered in section 3.

In the case of equation (4.1), the homogeneous linear equations for $k \geqslant 1$ are of Lamé type:

$$\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} - (h + n(n+1)\mathcal{P}(z))w = 0$$

where $\mathcal{P}$ is the elliptic Weierstrass function with poles at $2m\omega + 2m'\omega'$, for the choice $n = 1, h = 0$.

From the Fuchsian theory we easily obtain the two independent solutions of the homogeneous equation:

$$w_1(z) = (z - 2m\omega - 2m'\omega')^2 \mathcal{W}(z)$$

where $\mathcal{W}(z)$ is analytic in the domain of the point $2m\omega + 2m'\omega'$ and is different from zero at that point.

The second solution is

$$w_2(z) = c w_1(z) \int \frac{\mathrm{d}z'}{(w_1(z'))^2}.$$

The two solutions have, respectively, zeros of order 3 and poles of order 2. In our case, the solutions may also be expressed in the following way:

$$w_1 = \exp(-z\zeta(a)) \frac{\sigma(z+a)}{\sigma(z)}$$
$$w_2 = \exp(z\zeta(a)) \frac{\sigma(z-a)}{\sigma(z)}$$

where a is the solution of $\mathcal{P}(a) = h$.

Let us now consider what are the singularities of the complete solution of the inhomogeneous equations. We recall that in order to satisfy the necessary conditions for Painlevé integrability using the α-methods we have to require that $q^{(0)}$ is free from movable critical points and that the particular solutions of the inhomogeneous equations (4.17) are also free from movable critical points. In particular, in order to satisfy Painlevé conditions in our case we should impose that the particular solutions of (4.17) do not contain other singularities than double poles. But this cannot be true if $F(t)$ is not a trivial constant perturbation.

In fact, using the method of variation of constants, the particular solution may be written as

$$q_k^P(t) = -q_k^{(1)}(t) \int^t \frac{q_k^{(2)} \mathcal{S}_k}{\Delta} + q_k^{(2)}(t) \int^t \frac{q_k^{(1)} \mathcal{S}_k}{\Delta}$$

where $q_k^{(1)}$ and $q_k^{(2)}$ are two independent solutions of the homogeneous equation and

$$\Delta(t) = q_k^{(1)}(t)\dot{q}_k^{(2)}(t) - q_k^{(2)}(t)\dot{q}_k^{(1)}(t) = -3g_3$$

is the Wronskian. Then, by considering the leading behaviour of the solutions $q^{(1)}(t)$ and $q^{(2)}(t)$ we can show that a logarithmic singularity appears necessary at the level of the particular solution relative to the sixth equation which represents the analogue of the resonance condition of the local series expansions. This can be easily checked for the case of the separatrix solutions where we get

$$q_{(1)}^{(k)}(t) = (t - c_0)^4 \qquad q_{(2)}^{(k)}(t) = (t - c_0)^{-3} \qquad \Delta = -7.$$

Then the particular solutions of the inhomogeneous equations are

$$\begin{aligned}
q_p^{(1)}(t) &= 0 \\
q_p^{(2)}(t) &= \tfrac{1}{2}(\omega + \epsilon f_0) \\
q_p^{(3)}(t) &= \tfrac{1}{2}\epsilon f_1 t \\
q_p^{(4)}(t) &= \frac{(\omega + \epsilon f_0)^2}{40}(t - c_0)^2 + \frac{3\epsilon f_2}{5}t^2 - \frac{\epsilon f_2 c_0}{5}t + \frac{\epsilon f_2 c_0^2}{10} \\
q_p^{(5)}(t) &= \epsilon f_1(\omega + \epsilon f_0)\left[\frac{(t - c_0)^3}{12} + \frac{c_0}{20}(t - c_0)^2\right] \\
&\quad + \epsilon f_2\left[(t - c_0)^3 + \frac{9}{5}c_0(t - c_0)^2 + \frac{3}{2}c_0^2(t - c_0) + \frac{c_0^3}{2}\right] \\
q_p^{(6)}(t) &= -\frac{(t - c_0)^4}{7}\int^t \mathrm{d}\tau\left(\left[\frac{\epsilon f_2}{2}(\omega + \epsilon f_0) + \frac{\epsilon^2 f_1^2}{4}\right]\tau^2(\tau - c_0)^{-3} + 6\epsilon f_4\tau^4(\tau - c_0)^{-5}\right) \\
&\quad + \frac{(t - c_0)^{-3}}{7}\int^t \mathrm{d}\tau\left(\left[\frac{\epsilon f_2}{2}(\omega + \epsilon f_0) + \frac{\epsilon^2 f_1^2}{4}\right]\right. \\
&\quad \left. \times\tau^2(\tau - c_0)^2 + 6\epsilon f_4\tau^4(\tau - c_0)^2\right) \\
&= \frac{(t - c_0)^4}{7}\left[\frac{\epsilon f_2}{2}(\omega + \epsilon f_0) + \frac{\epsilon^2 f_1^2}{4} + 6\epsilon f_4\right]\log(t - c_0).
\end{aligned}$$

In the case of (4.2), $m = -\frac{1}{2}$ and $s = -\frac{5}{2}$ and $q^{(0)}(t) = \sqrt{\gamma}$ where γ is the elliptic function solution of the following differential equation:

$$\ddot{\gamma} - \frac{\dot{\gamma}^2}{2\gamma} - 2\gamma^3 = 0.$$

Along the separatrix the solution reduces to $\gamma(t) = \sqrt{\frac{3}{4}\frac{1}{(t-c_0)}}$. The linear homogeneous differential equations for $k \geqslant 1$ admit of the following solutions:

$$v_1(t) = \frac{t}{2}\frac{\dot{\gamma}}{\sqrt{\gamma}} + \frac{1}{2}\sqrt{\gamma} \qquad v_2(t) = \frac{\dot{\gamma}}{2\sqrt{\gamma}}$$

and have algebraic branching points of order $\frac{5}{2}$ and $-\frac{3}{2}$ respectively. Along the separatrix such homogeneous solutions reduce to $v_1(t) = (t - c_0)^{\frac{5}{2}}$ and $v_2(t) = (t - c_0)^{-\frac{3}{2}}$. As expected, logarithmic singularities appear from the particular solutions of the differential equation $\mathcal{F}^{(6)}$. Indeed, in the 'separatrix' case

$$q_p^{(6)} = \frac{(t - c_0)^{\frac{5}{2}}}{4}\left(\frac{3}{4}\right)^{\frac{1}{4}}\epsilon f_1 \log(t - c_0) + \cdots.$$

As before, in order to exclude the appearance of such logarithmic singularities in $q_p^{(6)}$, we should require that the perturbation $F(t)$ is trivially constant. Moreover, also in this case, such a solution as (A.2) is the first-order expansion term of the particular solution outside the separatrix and so there does follow that one expects logarithmic singularities in the solutions of the associated system of equations (4.7).

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